

Out-of-equilibrium dynamics of large- N ϕ^4 QFT in finite volume**C. Destri** ^(a,b) and **E. Manfredini** ^(b)*(a) Dipartimento di Fisica G. Occhialini,
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Abstract

The $\lambda\phi^4$ model in a finite volume is studied in the infinite N limit both at equilibrium and out of equilibrium, with particular attention to certain fundamental features of the broken symmetry phase. The numerical solution of the dynamical evolution equations shows that the zero-mode quantum fluctuations cannot grow macroscopically large starting from microscopic initial conditions. Thus we conclude that there is no evidence for a dynamical Bose-Einstein condensation, in the usual sense. On the other hand, out of equilibrium the long-wavelength fluctuations do scale with the linear size of the system, signalling dynamical infrared properties quite different from the equilibrium ones characteristic of the same approximation scheme.

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I. INTRODUCTION

In the last few years a great deal of attention has been paid to the study of interacting quantum fields out of equilibrium. There are, in fact, many interesting physical situations in which the standard S-matrix approach cannot give sensible information about the behavior of the system, because it evolves through a series of highly excited states (i.e., states of finite energy density).

As an example consider any model of cosmological inflation: it is not possible to extract precise predictions on physical observables without including in the treatment the quantum back-reaction of the field on the space-time geometry and on itself [1–3].

On the side of particle physics, the ultra-relativistic heavy-ion collisions, scheduled in the forthcoming years at CERN-SPS, BNL-RHIC and CERN-LHC, are supposed to produce hadron matter at very high densities and temperatures; in such a regime the usual approach based on particle scattering cannot be considered a good interpretative tool at all. To extract sensible information from the theory new computational schemes are necessary, that go beyond the simple Feynman diagram expansion. The use of resummation schemes, like the Hartree-Fock (HF) [4,5] approximation and the large N limit (LN) [6], or the Hard Thermal Loop resummation for systems at finite temperature (HTL) [7], can be considered a first step in this direction. They, in fact, enforce a sum over an infinite subset of Feynman diagrams that are dominant in a given region of the parameter space, where the simple truncation of the usual perturbative series at finite order cannot give sensible answers.

Quite recently HF and LN have been used in order to clarify some dynamical aspects of the large N ϕ^4 theory. For the reader's benefit and to better motivate this work, we give a very short summary of the conclusions reached in previous works: i) the early time evolution is dominated by a so-called “linear regime”, during which the energy initially stored in one (or few) modes of the field is transferred to other modes via either parametric or spinodal instabilities, resulting in a large particle production and a consequent dissipation for the initial condensate [8]; ii) the linear regime stops at a time scale $t_1 \propto \log(\lambda^{-1})$ (where λ is the quartic coupling constant), by which the effects of the quantum fluctuation become of the same order as the classical contribution and the dynamics turns completely non linear and non perturbative [8,9]; iii) after the time t_1 the relaxation occurs via power laws with anomalous dynamical exponent [9]; iv) the asymptotic particle distribution, obtained as the result of the copious particle production at the expenses of the “classical” energy, is strongly non-thermal [8,9]; and finally, v) at very large time scale, $t \sim \sqrt{V}$ (where V stands for the volume of the system), the non-perturbative and non-linear evolution might eventually produce the onset of a novel form of non-equilibrium Bose-Einstein condensation of the long-wavelength Goldstone bosons usually present in the broken symmetry phase of the model [9,10]. Another very interesting result in [10] concerns the dynamical Maxwell construction, which reproduces the flat region of the effective potential in case of broken symmetry as asymptotic fixed points of the background evolution. Moreover, the LN approximation scheme has been used to follow the evolution of a initial state characterized by an occupied spherical shell in momentum space (a spherical ‘tsunami’ [11]), around a particular momentum $|\vec{k}_0|$, with the following results: i) in a theory where the symmetry is spontaneously broken at zero density, if we start with a finite density initial state with restored symmetry, the spinodal instabilities lead to a dynamical symmetry breaking; ii)

the evolution produces a re-arrangement of the particle distribution towards low momenta, signalling the onset of Bose condensation; iii) the equation of state of the asymptotic gas is ultra-relativistic (even if the distribution is not thermal) [11].

In this article we present a detailed study, *in finite volume*, of dynamical evolution out of equilibrium for the Φ^4 scalar field in the large N limit. More precisely, we determine how such dynamics scales with the size of the periodic box containing the system in the case of uniform backgrounds. This is necessary to address questions like out-of-equilibrium symmetry breaking and dynamical Bose–Einstein condensation.

In section II we define the model in finite volume, giving all the relevant notations and definitions. We also stress the convexity of the effective potential as an exact result, valid for the full renormalized theory in any volume.

In section III we derive the large N approximation of the $O(N)$ –invariant version of $\lambda(\phi^2)^2$ model, according to the general rules of ref. [12]. In this derivation it appears evident the essential property of the $N \rightarrow \infty$ limit of being a particular type of *classical* limit, so that it leads to a classical phase space, a classical hamiltonian with associated Hamilton’s equations of motion [see eqs. (3.15), (3.16) and (3.17)]. We then minimize the hamiltonian function(al) and determine the conditions for massless Goldstone bosons (i.e. transverse fluctuations of the field) to form a Bose–Einstein condensate, delocalizing the vacuum field expectation value (see also ref. [13]). This necessarily requires that the width of the zero–mode fluctuations becomes macroscopically large, that is of the order of the volume. Only when the background takes one of the extremal values proper of symmetry breaking the width of the zero–mode fluctuations is of order $L^{1/2}$, as typical of a free massless spectrum.

The study of the lowest energy states of the model is needed for comparison with the results of the numerical simulations, which show that the zero–mode width σ_0 stays microscopic (that is such that $\sigma_0/\text{volume} \rightarrow 0$ when the volume diverges) whenever it starts from initial conditions in which it is microscopic. Our results, in fact, show clearly the presence of a time scale τ_L , proportional to the linear size L of the system, at which finite volume effects start to manifest. We shall give a very simple physical interpretation of this time scale in section IIID. The important point is that after τ_L the zero mode amplitude starts decreasing, then enters an erratic evolution, but never grows macroscopically large. This result is at odd with the interpretation of the linear late–time growth of the zero–mode width as a full dynamical Bose–Einstein condensation of Goldstone bosons, but is compatible with the “novel” form of BEC reported in [9–11].

In fact we do find that the size of the low–lying widths at time τ_L is of order L , to be compared to the equilibrium situation where they would be of order L^0 in the massive case or of order $L^{1/2}$ in the massless case. Perhaps the denomination “microscopic” should be reserved to this two possibilities. Therefore, since our initial condition are indeed microscopic in this restricted sense, we do observe in the out-of-equilibrium evolution a rapid transition to a different regime intermediate between the microscopic one and the macroscopic one characteristic of Bose–Einstein condensation. As we shall discuss more in detail later on, this fully agrees with the result found in [10], that the time–dependent field correlations vanish at large separations more slowly than for equilibrium free massless fields (as r^{-1} rather than r^{-2}), but definitely faster than the equilibrium broken symmetry phase characterized by constant correlations at large distances.

At any rate, when one considers microscopic initial conditions for the choice of bare

mass which corresponds to broken symmetry, the role itself of symmetry breaking is not very clear in the large N description of the out-of-equilibrium dynamics, making equally obscure the issues concerning the so-called quantum phase ordering [10]. This is because the limit $N \rightarrow \infty$ is completely saturated by gaussian states, which might signal the onset of symmetry breaking only developing macroscopically large fluctuations. Since such fluctuations do not appear to be there, the meaning itself of symmetry breaking, as something happening as times goes on and accompanied by some kind of phase ordering, is quite unclear. We postpone to a companion work [14] the discussion about the possibility of using more comprehensive approximation schemes, that include some non-gaussian features of the complete theory. As far as the large N approximation is concerned, we underline that an important limitation of our approach, as well as of those of the references mentioned above, is in any case the assumption of a uniform background. Nonetheless, phenomena like the asymptotic vanishing of the effective mass and the dynamical Maxwell construction, taking place in this contest of a uniform background and large N expansion, are certainly very significant manifestations of symmetry breaking and in particular of the Goldstone theorem which applies when a continuous symmetry is broken.

Finally, in section IV we summarize the results presented in this article and we sketch some interesting open problems that we plan to study in forthcoming works.

II. CUTOFF FIELD THEORY

We consider the N -component scalar field operator ϕ in a D -dimensional periodic box of size L and write its Fourier expansion as customary

$$\phi(x) = L^{-D/2} \sum_k \phi_k e^{ik \cdot x}, \quad \phi_k^\dagger = \phi_{-k}$$

with the wavevectors k naturally quantized: $k = (2\pi/L)n$, $n \in \mathbb{Z}^D$. The canonically conjugated momentum π has a similar expansion

$$\pi(x) = L^{-D/2} \sum_k \pi_k e^{ik \cdot x}, \quad \pi_k^\dagger = \pi_{-k}$$

with the commutation rules $[\phi_k^\alpha, \pi_{-k'}^\beta] = i \delta_{kk'}^{(D)} \delta^{\alpha\beta}$. The introduction of a finite volume should be regarded as a regularization of the infrared properties of the model, which allows to “count” the different field modes and is needed especially in the case of broken symmetry. In fact, all the results we have summarized in section I, have been obtained simulating the system directly in infinite volume, where the evolution equations contain momentum integrals, that must be computed numerically by a proper, but nonetheless rather arbitrary, discretization in momentum space. Of course, the final result should be as insensitive as possible to the particular choice of the integration grid. In such a situation, the definition of a “zero” mode and the interpretation of its late time behavior might not be rigorous enough, unless, for some reason, it turns out that a particular mode requires a different treatment compared to the others. In order to understand this point, it is necessary to put the system in a finite volume (a box of size V); the periodic boundary conditions let us single out the zero mode in a rigorous way and thus we can carefully analyze its scaling properties with V and get some information on the infinite volume limit.

To regularize also the ultraviolet behavior, we restrict the sums over wavevectors to the points lying within the D -dimensional sphere of radius Λ , that is $k^2 \leq \Lambda^2$, with $\mathcal{N} = \Lambda L/2\pi$ some large integer. Clearly we have reduced the original field-theoretical problem to a quantum-mechanical framework with finitely many (of order \mathcal{N}^{D-1}) degrees of freedom.

The ϕ^4 Hamiltonian reads

$$H = \frac{1}{2} \int d^D x \left[\boldsymbol{\pi}^2 + (\partial\phi)^2 + m_b^2 \phi^2 + \lambda_b (\phi^2)^2 \right] = \frac{1}{2} \sum_k \left[\boldsymbol{\pi}_k \cdot \boldsymbol{\pi}_{-k} + (k^2 + m_b^2) \phi_k \cdot \phi_{-k} \right] + \frac{\lambda_b}{4L^D} \sum_{k_1, k_2, k_3, k_4} (\phi_{k_1} \cdot \phi_{k_2})(\phi_{k_3} \cdot \phi_{k_4}) \delta_{k_1+k_2+k_3+k_4, 0}^{(D)} \quad (2.1)$$

where m_b^2 and λ_b should depend on the UV cutoff Λ in such a way to guarantee a finite limit $\Lambda \rightarrow \infty$ for all observable quantities. As is known [8,15], this implies triviality (that is vanishing of renormalized vertex functions with more than two external lines) for $D > 3$ and very likely also for $D = 3$. In the latter case triviality is manifest in the one-loop approximation and in large- N limit due to the Landau pole. For this reason we shall keep Λ finite and regard the ϕ^4 model as an effective low-energy theory (here low-energy means practically all energies below Planck's scale, due to the large value of the Landau pole for renormalized coupling constants of order one or less).

We shall work in the wavefunction representation where $\langle \varphi | \Psi \rangle = \Psi(\varphi)$ and

$$(\phi_0 \Psi)(\varphi) = \varphi_0 \Psi(\varphi), \quad (\pi_0 \Psi)(\varphi) = -i \frac{\partial}{\partial \varphi_0} \Psi(\varphi)$$

while for $k > 0$ (in lexicographic sense)

$$(\phi_{\pm k} \Psi)(\varphi) = \frac{1}{\sqrt{2}} (\varphi_k \pm i \varphi_{-k}) \Psi(\varphi), \quad (\pi_{\pm k} \Psi)(\varphi) = \frac{1}{\sqrt{2}} \left(-i \frac{\partial}{\partial \varphi_k} \pm \frac{\partial}{\partial \varphi_{-k}} \right) \Psi(\varphi)$$

Notice that by construction the variables φ_k are all real. Of course, when either one of the cutoffs are removed, the wave function $\Psi(\varphi)$ acquires infinitely many arguments and becomes what is usually called a *wavefunctional*.

In practice, the problem of studying the dynamics of the ϕ^4 field out of equilibrium consists now in trying to solve the time-dependent Schroedinger equation given an initial wavefunction $\Psi(\varphi, t = 0)$ that describes a state of the field far away from the vacuum. By this we mean a non-stationary state that, in the infinite volume limit $L \rightarrow \infty$, would lay outside the particle Fock space constructed upon the vacuum. This approach could be generalized in a straightforward way to mixtures described by density matrices, as done, for instance, in [16–18]. Here we shall restrict to pure states, for sake of simplicity and because all relevant aspects of the problem are already present in this case.

It is by now well known [8] that perturbation theory is not suitable for the purpose stated above. Due to parametric resonances and/or spinodal instabilities there are modes of the field that grow exponentially in time until they produce non-perturbative effects for any coupling constant, no matter how small. On the other hand, only few, by now standard, approximate non-perturbative schemes are available for the ϕ^4 theory, and to these we have to resort after all. We shall consider here only the large N expansion to leading order, remanding to another work the definition of a time-dependent Hartree-Fock

(tdHF) approach [14] (a generalization of the treatment given, for instance, in [4]). In fact these two methods are very closely related, as shown in [19], where several techniques to derive reasonable dynamical evolution equations for non-equilibrium ϕ^4 are compared.

We close this section by stressing that the introduction of both a UV and IR cutoff allows to easily derive the well-known rigorous result concerning the flatness of the effective potential. In fact $V_{\text{eff}}(\bar{\phi})$ is a convex analytic function in a finite neighborhood of $\bar{\phi} = 0$, as long as the cutoffs are present, due to the uniqueness of the ground state. In the infrared limit $L \rightarrow \infty$, however, $V_{\text{eff}}(\bar{\phi})$ might flatten around $\bar{\phi} = 0$. Of course this possibility would apply in case of spontaneous symmetry breaking, that is for a double-well classical potential. This is a subtle and important point that will play a crucial role later on, even if the effective potential is relevant for the static properties of the model rather than the dynamical evolution out of equilibrium that interests us here. In fact, the dynamical evolution in QFT is governed by the CTP effective action [20,21] and one might expect that, although non-local in time, it asymptotically reduces to a multiple of the effective potential for trajectories of $\bar{\phi}(t)$ with a fixed point at infinite time. In such case there should exist a one-to-one correspondence between fixed points and minima of the effective potential.

III. LARGE N EXPANSION AT LEADING ORDER

A. Definitions

In this section we consider the standard non-perturbative approach to the ϕ^4 model which is applicable also out of equilibrium, namely the large N method as presented in [22]. However we shall follow a different derivation which makes the gaussian nature of the $N \rightarrow \infty$ limit more explicit.

It is known that the theory described by the Hamiltonian (2.1) is well behaved for large N , provided that the quartic coupling constant λ_b is rescaled with $1/N$. For example, it is possible to define a perturbation theory, based on the small expansion parameter $1/N$, in the framework of which one can compute any quantity at any chosen order in $1/N$. From the diagrammatic point of view, this procedure corresponds to a resummation of the usual perturbative series that automatically collects all the graphs of a given order in $1/N$ together [6]. Moreover, it has been established since the early 80's that the leading order approximation (that is the strict limit $N \rightarrow \infty$) is actually a classical limit [12], in the sense that there exists a classical system (i.e., a classical phase space, a Poisson bracket and a classical Hamiltonian) whose dynamics controls the evolution of all fundamental quantum observables, such as field correlation functions, in the $N \rightarrow \infty$ limit. For instance, from the absolute minimum of the classical Hamiltonian one reads the energy of the ground state, while the spectrum is given by the frequencies of small oscillations about this minimum, etc. etc.. We are here interested in finding an efficient and rapid way to compute the quantum evolution equations for some observables in the $N \rightarrow \infty$ limit, and we will see that this task is easily accomplished just by deriving the canonical Hamilton equations from the large N classical Hamiltonian.

Following Yaffe [12], we write the quantum mechanical hamiltonian as

$$H = Nh(A, C) \tag{3.1}$$

in terms of the square matrices A , C with operator entries (ϖ_k is the canonical momentum conjugated to the real mode φ_k)

$$A_{kk'} = \frac{1}{N} \varphi_k \cdot \varphi_{k'} , \quad C_{kk'} = \frac{1}{N} \varpi_k \cdot \varpi_{k'}$$

These are example of “classical” operators, whose two-point correlation functions factorize in the $N \rightarrow \infty$ limit. This can be shown by considering the coherent states

$$\Psi_{z,q,p}(\varphi) = C(z) \exp \left[i \sum_k \mathbf{p}_k \cdot \varphi_k - \frac{1}{2N} \sum_{kk'} z_{kk'} (\varphi_k - \mathbf{q}_k) \cdot (\varphi_{k'} - \mathbf{q}_{k'}) \right] \quad (3.2)$$

where the complex symmetric matrix z has a positive definite real part while \mathbf{p}_k and \mathbf{q}_k are real and coincide, respectively, with the coherent state expectation values of ϖ_k and φ_k . As these parameters take all their possible values, the coherent states form an overcomplete set in the cutoff Hilbert space of the model. The crucial property which ensures factorization is that they become all orthogonal in the $N \rightarrow \infty$ limit. Moreover one can show [12] that the coherent states parameters form a classical phase space with Poisson brackets

$$\{q_k^i, p_{k'}^j\}_{\text{P.B.}} = \delta_{kk'} \delta^{ij} , \quad \{w_{kk'}, v_{qq'}\}_{\text{P.B.}} = \delta_{kq} \delta_{k'q'} + \delta_{kq'} \delta_{k'q}$$

where w and v reparametrize z as $z = \frac{1}{2}w^{-1} + i v$. It is understood that the dimensionality of the vectors \mathbf{q}_k and \mathbf{p}_k is arbitrary but finite [that is, only a finite number, say n , of pairs $(\varphi_k^i, \varpi_k^i)$ may take a nonvanishing expectation value as $N \rightarrow \infty$].

Once applied to the classical operators $A_{kk'}$ and $C_{kk'}$ the large N factorization allow to obtain the classical hamiltonian by simply replacing A and C in eq. (3.1) by the coherent expectation values

$$\langle A_{kk'} \rangle = \mathbf{q}_k \cdot \mathbf{q}_{k'} + w_{kk'} , \quad \langle C_{kk'} \rangle = \mathbf{p}_k \cdot \mathbf{p}_{k'} + (v w v)_{kk'} + \frac{1}{4}(w^{-1})_{kk'}$$

In our situation, having assumed a uniform background expectation value for ϕ , we have $\mathbf{q}_k = \mathbf{p}_k = 0$ for all $k \neq 0$; moreover, translation invariance implies that w and v are diagonal matrices, so that we may set

$$w_{kk'} = \sigma_k^2 \delta_{kk'} , \quad v_{kk'} = \frac{s_k}{\sigma_k} \delta_{kk'}$$

in term of the canonical couples (σ_k, s_k) which satisfy $\{\sigma_k, s_{k'}\}_{\text{P.B.}} = \delta_{kk'}$. Notice that the σ_k are just the widths (rescaled by $N^{-1/2}$) of the $O(N)$ symmetric and translation invariant gaussian coherent states.

Thus we find the classical hamiltonian

$$h_{\text{cl}} = \frac{1}{2}(\mathbf{p}_0^2 + m_b^2 \mathbf{q}_0^2) + \frac{1}{2} \sum_k \left[s_k^2 + (k^2 + m_b^2) \sigma_k^2 + \frac{1}{4\sigma_k^2} \right] + \frac{\lambda_b}{4L^D} \left(\mathbf{q}_0^2 + \sum_k \sigma_k^2 \right)^2$$

where by Hamilton's equations of motion $\mathbf{p}_0 = \dot{\mathbf{q}}_0$ and $s_k = \dot{\sigma}_k$. The corresponding conserved energy density $\mathcal{E} = L^{-D} h_{\text{cl}}$ may be written

$$\begin{aligned} \mathcal{E} &= \mathcal{T} + \mathcal{V} , \quad \mathcal{T} = \frac{1}{2} \dot{\bar{\phi}}^2 + \frac{1}{2L^D} \sum_k \dot{\sigma}_k^2 \\ \mathcal{V} &= \frac{1}{2L^D} \sum_k \left(k^2 \sigma_k^2 + \frac{1}{4\sigma_k^2} \right) + V(\bar{\phi}^2 + \Sigma) , \quad \Sigma = \frac{1}{L^D} \sum_k \sigma_k^2 \end{aligned} \quad (3.3)$$

where $\bar{\phi} = L^{-D/2} \mathbf{q}_0$ and V is the $O(N)$ -invariant quartic potential regarded as a function of ϕ^2 , that is $V(z) = \frac{1}{2} m_b^2 z + \frac{1}{4} \lambda_b z^2$. It is worth noticing that eq. (3.3) would apply as is to generic $V(z)$.

B. Static properties

Let us consider first the static aspects embodied in the effective potential $V_{\text{eff}}(\bar{\phi})$, that is the minimum of the potential energy \mathcal{V} at fixed $\bar{\phi}$. We first define in a precise way the unbroken symmetry phase, in this large N context, as the case when $V_{\text{eff}}(\bar{\phi})$ has a unique minimum at $\bar{\phi} = 0$ in the limit of infinite volume. Minimizing \mathcal{V} w.r.t. σ_k yields

$$\begin{aligned} \sigma_k^2 &= \frac{1}{2\sqrt{k^2 + M^2}}, \quad M^2 = m_b^2 + 2V'(\bar{\phi}^2 + \Sigma) \\ &= m_b^2 + \lambda_b \bar{\phi}^2 + \frac{\lambda_b}{L^D} \sum_k \frac{1}{2\sqrt{k^2 + M^2}} \end{aligned} \quad (3.4)$$

that is the widths characteristic of a free theory with self-consistent mass M fixed by the gap equation. By the assumption of unbroken symmetry, when $\bar{\phi} = 0$ and at infinite volume M coincides with the equilibrium mass m of the theory, that may be regarded as independent scale parameter. Since in the limit $L \rightarrow \infty$ sums are replaced by integrals

$$\Sigma \rightarrow \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \sigma_k^2$$

we obtain the standard bare mass parameterization

$$m_b^2 = m^2 - \lambda_b I_D(m^2, \Lambda), \quad I_D(z, \Lambda) \equiv \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \frac{1}{2\sqrt{k^2 + z}} \quad (3.5)$$

and the renormalized gap equation

$$M^2 = m^2 + \lambda \bar{\phi}^2 + \lambda \left[I_D(M^2, \Lambda) - I_D(m^2, \Lambda) \right]_{\text{finite}} \quad (3.6)$$

which implies, when $D = 3$,

$$\lambda_b = \lambda \left(1 - \frac{\lambda}{8\pi^2} \log \frac{2\Lambda}{m\sqrt{e}} \right)^{-1} \quad (3.7)$$

with a suitable choice of the finite part. No coupling constant renormalization occurs instead when $D = 1$. The renormalized gap equation (3.6) may also be written quite concisely

$$\frac{M^2}{\hat{\lambda}(M)} = \frac{m^2}{\hat{\lambda}(m)} + \bar{\phi}^2 \quad (3.8)$$

in terms of the one-loop running couplings constant

$$\hat{\lambda}(\mu) = \lambda \left[1 - \frac{\lambda}{8\pi^2} \log \frac{\mu}{m} \right]^{-1}, \quad \hat{\lambda}(m) = \lambda, \quad \hat{\lambda}(2\Lambda e^{-1/2}) = \lambda_b$$

It is the Landau pole in $\hat{\lambda}(2\Lambda e^{-1/2})$ that actually forbids the limit $\Lambda \rightarrow \infty$. Hence we must keep the cutoff finite and smaller than Λ_{pole} , so that the theory does retain a slight inverse-power dependence on it. At any rate, there exists a very wide window where this dependence is indeed very weak for couplings of order one or less, since $\Lambda_{\text{pole}} = \frac{1}{2}m \exp(1/2 + 8\pi^2/\lambda) \gg m$. Moreover, we see from eq. (3.8) that for $\sqrt{\lambda}|\bar{\phi}|$ much smaller than the Landau pole there are two solutions for M , one “physical”, always larger than m and of the same order of $m + \sqrt{\lambda}|\bar{\phi}|$, and one “unphysical”, close to the Landau pole.

One can now easily verify that the effective potential has indeed a unique minimum in $\bar{\phi} = 0$, as required. In fact, if we assign arbitrary $\bar{\phi}$ -dependent values to the widths σ_k , (minus) the effective force reads

$$\frac{d}{d\bar{\phi}^i} \mathcal{V}(\bar{\phi}, \{\sigma_k(\bar{\phi})\}) = M^2 \bar{\phi}^i + \sum_k \frac{\partial \mathcal{V}}{\partial \sigma_k} \frac{d\sigma_k}{d\bar{\phi}^i} \quad (3.9)$$

and reduces to $M^2 \bar{\phi}^i$ when the widths are extremal as in eq. (3.4); but M^2 is positive for unbroken symmetry and so $\bar{\phi} = 0$ is the unique minimum.

We define the symmetry as broken whenever the infinite volume V_{eff} has more than one minimum. Of course, as long as L is finite, V_{eff} has a unique minimum in $\bar{\phi} = 0$, because of the uniqueness of the ground state in Quantum Mechanics, as already discussed in section II. Let us therefore proceed more formally and take the limit $L \rightarrow \infty$ directly on the potential energy \mathcal{V} . It reads

$$\mathcal{V} = \frac{1}{2} \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \left(k^2 \sigma_k^2 + \frac{1}{4\sigma_k^2} \right) + V(\bar{\phi}^2 + \Sigma), \quad \Sigma = \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \sigma_k^2$$

where we write for convenience the tree-level potential V in the positive definite form $V(z) = \frac{1}{4}\lambda_b(z + m_b^2/\lambda_b)^2$. \mathcal{V} is now the sum of two positive definite terms. Suppose there exists a configuration such that $V(\bar{\phi}^2 + \Sigma) = 0$ and the first term in \mathcal{V} is at its minimum. Then this is certainly the absolute minimum of \mathcal{V} . This configuration indeed exists at infinite volume when $D = 3$:

$$\sigma_k^2 = \frac{1}{2|k|}, \quad \bar{\phi}^2 = v^2, \quad m_b^2 = -\lambda_b [v^2 + I_3(0, \Lambda)] \quad (3.10)$$

where the nonnegative v should be regarded as an independent parameter fixing the scale of the symmetry breaking. It replaces the mass parameter m of the unbroken symmetry case: now the theory is massless in accordance with Goldstone theorem. On the contrary, if $D = 1$ this configuration is not allowed due to the infrared divergences caused by the massless nature of the width spectrum. This is just the standard manifestation of Mermin–Wagner–Coleman theorem that forbids continuous symmetry breaking in a two-dimensional space–time [23].

At finite volumes we cannot minimize the first term in \mathcal{V} since this requires σ_0 to diverge, making it impossible to keep $V(\bar{\phi}^2 + \Sigma) = 0$. In fact we know that the uniqueness of the ground state with finitely many degrees of freedom implies the minimization equations (3.4) to hold always true with a M^2 strictly positive. Therefore, broken symmetry should manifest itself as the situation in which the equilibrium value of M^2 is a positive definite function of L which vanishes in the $L \rightarrow \infty$ limit.

We can confirm this qualitative conclusion as follows. We assume that the bare mass has the form given in eq. (3.10) and that $\bar{\phi}^2 = v^2$ too. Minimizing the potential energy leads always to the massive spectrum, eq. (3.4), with the gap equation

$$\frac{M^2}{\lambda_b} = \frac{1}{2L^3 M} + \frac{1}{2L^3} \sum_{k \neq 0} \frac{1}{\sqrt{k^2 + M^2}} - \frac{\Lambda^2}{8\pi^2} \quad (3.11)$$

If $M^2 > 0$ does not vanish too fast for large volumes, or stays even finite, then the sum on the modes has a behavior similar to the corresponding infinite volume integral: there is a quadratic divergence that cancels the infinite volume contribution, and a logarithmic one that renormalizes the bare coupling. The direct computation of the integral would produce a term containing the $M^2 \log(\Lambda/M)$. This can be split into $M^2 [\log(\Lambda/v) - \log(M/v)]$ by using v as mass scale. The first term renormalizes the coupling correctly, while the second one vanishes if M^2 vanishes in the infinite volume limit.

When $L \rightarrow \infty$, the asymptotic solution of (3.11) reads

$$M = \left(\frac{\lambda}{2}\right)^{1/3} L^{-1} + \text{h.o.t.}$$

that indeed vanishes in the limit. Note also that the exponent is consistent with the assumption made above that M vanishes slowly enough to approximate the sum over $k \neq 0$ with an integral with the same M .

Let us now consider a state whose field expectation value $\bar{\phi}^2$ is different from v^2 . If $\bar{\phi}^2 > v^2$, the minimization equations (3.4) leads to a positive squared mass spectrum for the fluctuations, with M^2 given self-consistently by the gap equation. On the contrary, as soon as $\bar{\phi}^2 < v^2$, one immediately see that a positive M^2 cannot solve the gap equation

$$M^2 = \lambda_b \left(\bar{\phi}^2 - v^2 + \frac{\sigma_0^2}{L^3} + \frac{1}{2L^3} \sum_{k \neq 0} \frac{1}{\sqrt{k^2 + M^2}} - \frac{\Lambda^2}{8\pi^2} \right)$$

if we insist on the requirement that σ_0 not be macroscopic. In fact, the r.h.s. of the previous equation is negative, no matter which positive value for the effective mass we choose, at least for L large enough. But nothing prevent us to consider a static configuration for which the amplitude of the zero mode is macroscopically large (i.e. it rescales with the volume L^3). Actually, if we choose

$$\frac{\sigma_0^2}{L^3} = v^2 - \bar{\phi}^2 + \frac{1}{2L^3 M}$$

we obtain the same equation as we did before and the same value for the potential, that is the minimum, in the limit $L \rightarrow \infty$. Note that at this level the effective mass M needs not to have the same behavior in the $L \rightarrow \infty$ limit, but it is free of rescaling with a different power of L . We can be even more precise: we isolate the part of the potential that refers to the zero mode width σ_0 (Σ' does not contain the σ_0 contribution)

$$\frac{1}{2} \left[m_b^2 + \lambda_b (\bar{\phi}^2 + \Sigma') \right] \frac{\sigma_0^2}{L^3} + \frac{\lambda_b}{4} \frac{\sigma_0^4}{L^6} + \frac{1}{8L^3 \sigma_0^2}$$

and we minimize it, keeping $\bar{\phi}^2$ fixed. The minimum is attained at $t = \sigma_0^2/L^3$ solution of the cubic equation

$$\lambda_b t^3 + \alpha \lambda_b t^2 - \frac{1}{4} L^{-6} = 0$$

where $\alpha = \bar{\phi}^2 - v^2 + \Sigma' - I_3(0, \Lambda)$. Note that $\lambda_b \alpha$ depends on L and it has a finite limit in infinite volume: $\lambda(\bar{\phi}^2 - v^2)$. The solution of the cubic equation is

$$\lambda_b t = \lambda_b(v^2 - \bar{\phi}^2) + \frac{1}{4}[L^3(v^2 - \bar{\phi}^2)]^{-2} + \text{h.o.t.}$$

from which the effective mass can be identified as proportional to L^{-3} . The stability equations for all the other modes can now be solved by a massive spectrum, in a much similar way as before.

Since σ_0 is now macroscopically large, the infinite volume limit of the σ_k distribution (that gives a measure of the *transverse* fluctuations in the $O(N)$ model) develop a δ -like singularity, signalling a Bose condensation of the Goldstone bosons:

$$\sigma_k^2 = (v^2 - \bar{\phi}^2) \delta^{(D)}(k) + \frac{1}{2k} \quad (3.12)$$

At the same time it is evident that the minimal potential energy is the same as when $\bar{\phi}^2 = v^2$, that is the effective potential flattens, in accord with the Maxwell construction.

Eq. (3.12) corresponds in configuration space to the 2-point correlation function

$$\lim_{N \rightarrow \infty} \frac{\langle \phi(x) \cdot \phi(y) \rangle}{N} = \bar{\phi}^2 + \int \frac{d^D k}{(2\pi)^D} \sigma_k^2 e^{ik \cdot (x-y)} = C(\bar{\phi}^2) + \Delta_D(x-y) \quad (3.13)$$

where $\Delta_D(x-y)$ is the massless free-field equal-time correlator, while

$$C(\bar{\phi}^2) = v^2 \Theta(v^2 - \bar{\phi}^2) + \bar{\phi}^2 \Theta(\bar{\phi}^2 - v^2) = \max(v^2, \bar{\phi}^2) \quad (3.14)$$

This expression can be extended to unbroken symmetry by setting in that case $C(\bar{\phi}^2) = \bar{\phi}^2$.

Quite evidently, when eq. (3.14) holds, symmetry breaking can be inferred from the limit $|x-y| \rightarrow \infty$, if clustering is assumed [24,25], since $\Delta_D(x-y)$ vanishes for large separations. Of course this contradicts the infinite volume limit of the finite-volume definition, $\bar{\phi} = \lim_{N \rightarrow \infty} N^{-1/2} \langle \phi(x) \rangle$, except at the extremal points $\bar{\phi}^2 = v^2$.

In fact the $L \rightarrow \infty$ limit of the finite volume states with $\bar{\phi}^2 < v^2$ do violate clustering, because they are linear superpositions of vectors belonging to superselected sectors and therefore they are indistinguishable from statistical mixtures. We can give the following intuitive picture for large N . Consider any one of the superselected sectors based on a physical vacuum with $\bar{\phi}^2 = v^2$. By condensing a macroscopic number of transverse Goldstone bosons at zero-momentum, one can build coherent states with rotated $\bar{\phi}$. By incoherently averaging over such rotated states one obtains new states with field expectation values shorter than v by any prefixed amount. In the large N approximation this averaging is necessarily uniform and is forced upon us by the residual $O(N-1)$ symmetry.

C. Out-of-equilibrium dynamics

We now turn to the dynamics out of equilibrium in this large N context. It is governed by the equations of motion derived from the total energy density \mathcal{E} in eq. (3.3), that is

$$\frac{d^2 \bar{\phi}}{dt^2} = -M^2 \bar{\phi}, \quad \frac{d^2 \sigma_k}{dt^2} = -(k^2 + M^2) \sigma_k + \frac{1}{4\sigma_k^3} \quad (3.15)$$

where the generally time-dependent effective squared mass M^2 is given by

$$M^2 = m^2 + \lambda_b \left[\bar{\phi}^2 + \Sigma - I_D(m^2, \Lambda) \right] \quad (3.16)$$

in case of unbroken symmetry and

$$M^2 = \lambda_b \left[\bar{\phi}^2 - v^2 + \Sigma - I_3(0, \Lambda) \right] \quad (3.17)$$

for broken symmetry in $D = 3$.

At time zero, the specific choice of initial conditions for σ_k that give the smallest energy contribution, that is

$$\dot{\sigma}_k = 0, \quad \sigma_k^2 = \frac{1}{2\sqrt{k^2 + M^2}} \quad (3.18)$$

turns eq. (3.16) into the usual gap equation (3.4). For any value of $\bar{\phi}$ this equation has one solution smoothly connected to the value $M = m$ at $\bar{\phi} = 0$. Of course other initial conditions are possible. The only requirement is that the corresponding energy must differ from that of the ground state by an ultraviolet finite amount, as it occurs for the choice (3.18). In fact this is guaranteed by the gap equation itself, as evident from eq. (3.9): when the widths σ_k are extremal the effective force is finite, and therefore so are all potential energy differences.

This simple argument needs a refinement in two respects.

Firstly, in case of symmetry breaking the formal energy minimization w.r.t. σ_k leads always to eqs. (3.18), but these are acceptable initial conditions only if the gap equation that follows from eq. (3.17) in the $L \rightarrow \infty$ limit, namely

$$M^2 = \lambda_b \left[\bar{\phi}^2 - v^2 + I_D(M^2, \Lambda) - I_D(0, \Lambda) \right] \quad (3.19)$$

admits a nonnegative, physical solution for M^2 .

Secondly, ultraviolet finiteness only requires that the sum over k in eq. (3.9) be finite and this follows if eq. (3.18) holds at least for k large enough, solving the issue raised in the first point: negative M^2 are allowed by imposing a new form of gap equation

$$M^2 = \lambda_b \left[\bar{\phi}^2 - v^2 + \frac{1}{L^D} \sum_{k^2 < |M^2|} \sigma_k^2 + \frac{1}{L^D} \sum_{k^2 > |M^2|} \frac{1}{2\sqrt{k^2 - |M^2|}} - I_D(0, \Lambda) \right] \quad (3.20)$$

where all σ_k with $k^2 < |M^2|$ are kept free (but all by hypothesis microscopic) initial conditions. Of course there is no energy minimization in this case. To determine when this new

form is required, we observe that, neglecting the inverse-power corrections in the UV cutoff we may write eq. (3.19) in the following form

$$\frac{M^2}{\hat{\lambda}(M)} = \bar{\phi}^2 - v^2 \quad (3.21)$$

There exists a positive solution M^2 smoothly connected to the ground state, $\bar{\phi}^2 = v^2$ and $M^2 = 0$, only provided $\bar{\phi}^2 \geq v^2$. So, in the large N limit, as soon as we start with $\bar{\phi}^2 \leq v^2$, we cannot satisfy the gap equation with a positive value of M^2 .

Once a definite choice of initial conditions is made, the system of differential equations (3.15), (3.16) or (3.17) can be solved numerically with standard integration algorithms. This has been already done by several authors [8–10], working directly in infinite volume, with the following general results. In the case of unbroken symmetry it has been established that the σ_k corresponding to wavevectors k in the so-called forbidden bands with parametric resonances grow exponentially in time until their growth is shut off by the back-reaction. For broken symmetry it is the region in k -space with the spinodal instabilities caused by an initially negative M^2 , whose widths grow exponentially before the back-reaction shutoff. After the shutoff time the effective mass tends to a positive constant for unbroken symmetry and to zero for broken symmetry (in $D=3$), so that the only width with a chance to keep growing indefinitely is σ_0 for broken symmetry.

Of course, in all these approaches the integration over modes in the back-reaction Σ cannot be done exactly and is always replaced by a discrete sum of a certain type, depending on the details of the algorithms. Hence there exists always an effective infrared cutoff, albeit too small to be detectable in the numerical outputs. A possible troublesome aspect of this is the proper identification of the zero-mode width σ_0 . Even if a (rather arbitrary) choice of discretization is made where a σ_0 appears, it is not really possible to determine whether during the exponential growth or after such width becomes of the order of the volume. Our aim is just to answer this question and therefore we perform our numerical evolution in finite volumes of several growing sizes. Remanding to the appendix for the details of our method, we summarize our results in the next subsection.

D. Numerical results

After a careful study in $D = 3$ of the scaling behavior of the dynamics with respect to different values of L , the linear size of the system, we reached the following conclusion: there exist a L -dependent time, that we denote by τ_L , that splits the evolution in two parts; for $t \leq \tau_L$, the behavior of the system does not differ appreciably from its counterpart at infinite volume, while finite volume effects abruptly alter the evolution as soon as t exceeds τ_L ; moreover

- τ_L is proportional to the linear size of the box L and so it rescales as the cubic root of the volume.
- τ_L does not depend on the value of the quartic coupling constant λ , at least in a first approximation.

The figures show the behavior of the width of the zero mode σ_0 (see Fig. 1), of the squared effective mass M^2 (see Fig. 2) and of the back-reaction Σ (see Fig. 3), in the more interesting case of broken symmetry. The initial conditions are chosen in several different ways (see the appendix for details), but correspond to a negative M^2 at early times with the initial widths all microscopic, that is at most of order $L^{1/2}$. This is particularly relevant for the zero-mode width σ_0 , which is instead macroscopic in the lowest energy state when $\bar{\phi}^2 < v^2$, as discussed above. As for the background, the figures are relative to the simplest case $\bar{\phi} = 0 = \dot{\bar{\phi}}$, but we have considered also initial conditions with $\bar{\phi} > 0$, reproducing the “dynamical Maxwell construction” observed in ref. [10]. At any rate, for the purposes of this work, above all it is important to observe that, due to the quantum back-reaction, M^2 rapidly becomes positive, within the so-called *spinodal time* [8–10], and then, for times before τ_L , the *weakly dissipative* regime takes place where M^2 oscillates around zero with amplitude decreasing as t^{-1} and a frequency fixed by the largest spinodal wavevector, in complete agreement with the infinite-volume results [10]. Correspondingly, after the exponential growth until the spinodal time, the width of the zero-mode grows on average linearly with time, reaching a maximum for $t \simeq \tau_L$. Precisely, σ_0 performs small amplitude oscillations with the same frequency of M^2 around a linear function of the form $A + Bt$, where $A, B \approx \lambda^{-1/2}$ (see Fig. 4), confirming what already found in refs. [9,10]; then quite suddenly it turns down and enters long irregular Poincaré-like cycles. Since the spinodal oscillation frequency does not depend appreciably on L , the curves of σ_0 at different values of L are practically identical for $t < \tau_L$. After a certain number of complete oscillations, a number that scales linearly with L , a small change in the behavior of M^2 (see Fig. 5) determines an inversion in σ_0 (see Fig. 6), evidently because of a phase crossover between the two oscillation patterns. Shortly after τ_L dissipation practically stops as the oscillations of M^2 stop decreasing in amplitude and become more and more irregular, reflecting the same irregularity in the evolution of the widths.

We can give a straightforward physical interpretation for the presence of the time scale τ_L . As shown in [10], long after the spinodal time t_1 , the effective mass oscillates around zero with a decreasing amplitude and affects the quantum fluctuations in such a way that the equal-time two-point correlation function contains a time-dependent non-perturbative disturbance growing at twice the speed of light. This is interpreted in terms of large numbers of Goldstone bosons equally produced at any point in space (due to translation invariance) and radially propagating at the speed of light. This picture applies also at finite volumes, in the bulk, for volumes large enough. Hence, due to our periodic boundary conditions, after a time exactly equal to $L/2$ the forward wave front meets the backward wave front at the opposite point with respect to the source, and the propagating wave starts interfering with itself and heavily changes the dynamics with respect to that in infinite volume. This argument leads us to give the value of π for the proportionality coefficient between τ_L and $L/2\pi$, prevision very well verified by the numerical results, as can be inferred by a look at the figures.

The main consequence of this scenario is that the linear growth of the zero-mode width at infinite volume should not be interpreted as a standard form of Bose-Einstein Condensation (BEC), occurring with time, but should be consistently considered as “novel” form of dynamical BEC, as found by the authors of [10]. In fact, if a macroscopic condensation were really there, the zero mode would develop a δ function in infinite volume, that would

be announced by a width of the zero mode growing to values $O(L^{3/2})$ at any given size L . Now, while it is surely true that when we push L to infinity, also the time τ_L tends to infinity, allowing the zero mode to grow indefinitely, it is also true that, at any fixed though arbitrarily large volume, the zero mode never reaches a width $O(L^{3/2})$, just because $\tau_L \propto L$. In other words, if we start from initial conditions where σ_0 is microscopic, then it never becomes macroscopic later on.

On the other hand, looking at the behavior of the mode functions of momenta $k = (2\pi/L)n$ for n fixed but for different values of L , one realizes that they obey a scaling similar to that observed for the zero-mode: they oscillate in time with an amplitude and a period that are $O(L)$ (see fig. 7 and 8). Thus, each mode shows a behavior that is exactly half a way between a macroscopic amplitude [i.e. $O(L^{3/2})$] and a usual microscopic one [i.e. at most $O(L^{1/2})$]. This means that the spectrum of the quantum fluctuations at times of the order of the diverging volume can be interpreted as a *massless* spectrum of *interacting* Goldstone modes, because their power spectrum develops in the limit a $1/k^2$ singularity, rather than the $1/k$ pole typical of free massless modes. As a consequence the equal-time field correlation function [see eq. (3.13)] will fall off as $|\mathbf{x} - \mathbf{y}|^{-1}$ for large separations smaller only than the diverging elapsed time. This is in accord with what found in [10], where the same conclusion was reached after a study of the correlation function for the scalar field in infinite volume.

The fact that each mode never becomes macroscopic, if it started microscopic, might be regarded as a manifestation of unitarity in the large N approximation: an initial gaussian state with only microscopic widths satisfies clustering and clustering cannot be spoiled by a unitary time evolution. As a consequence, in the infinite-volume late-time dynamics, the zero-mode width σ_0 does not play any special role and only the behavior of σ_k as $k \rightarrow 0$ is relevant. As already stated above, it turns out from our numerics as well as from refs. [9–11] that this behavior is of a novel type characteristic both of the out-of-equilibrium dynamics and of the equilibrium finite-temperature theory [26], with $\sigma_k \propto 1/k$.

A final comment should be made about the periodic boundary conditions used for these simulations. This choice guarantees the translation invariance of the dynamics needed to consider a stable uniform background. If we had chosen other boundary conditions (Dirichlet or Neumann, for instance), the translation symmetry would have been broken and an uniform background would have become non-uniform pretty soon. Of course, we expect the bulk behavior to be independent of the particular choice for the boundary conditions in the infinite volume limit, even if a rigorous proof of this statement is still lacking.

IV. DISCUSSION AND PERSPECTIVES

In this work we have presented a rather detailed study of the dynamical evolution out of equilibrium, in finite volume (a cubic box of size L in 3D), for the ϕ^4 QFT in the large N limit. For comparison, we have also analyzed some static characteristics of the theory both in unbroken and broken symmetry phases.

We have reached the conclusion, based on strong numerical evidence, that the linear growth of the zero-mode quantum fluctuations, observed already in the large N approach of refs. [9–11], may be consistently interpreted as a “novel” form of dynamical Bose–Einstein

condensation, different from the traditional one in finite temperature field theory at equilibrium. In fact, in finite volume, σ_0 never grows to $O(L^{3/2})$ if it starts from a microscopic value, that is at most of order $L^{1/2}$. On the other hand all long-wavelength fluctuations rapidly become of order L , signalling a novel infrared behavior quite different from free massless fields at equilibrium [recall that the large N approximation is of mean field type, with no direct interaction among particle excitations]. This is in agreement with the properties of the two-point function determined in [10].

The numerical evidence for the linear dependence of τ_L on L is very strong, and the qualitative argument given in the previous section clearly explains the physics that determines it. Nonetheless a solid analytic understanding of the detailed (quantitative) mechanism that produces the inversion of $\dot{\sigma}_0$ around τ_L and its subsequent irregular behavior, is, at least in our opinion, more difficult to obtain. One could use intuitive and generic arguments like the quantization of momentum in multiples of $2\pi/L$, but the evolution equations do not have any simple scaling behavior towards a universal form, when mass dimensions are expressed in multiples of $2\pi/L$ and time in multiples of L . Moreover, the qualitative form of the evolution depends heavily on our choice of initial conditions. In fact, before finite volumes effects show up, the trajectories of the quantum modes are rather complex but regular enough, having a small-scale quasi-periodic almost mode-independent motion within a large-scale quasi-periodic mode-dependent envelope, with a very delicate resonant equilibrium [Cfr. Fig. 1 and 7]. Apparently [Cfr. Fig. 5 and 6], it is a sudden small beat that causes the turn around of the zero-mode and of the other low-lying modes (with many thousands of coupled modes, it is very difficult for the delicate resonant equilibrium to fully come back ever again), but we think that a deeper comprehension of the non-linear coupled dynamics is needed in order to venture into a true analytic explanation.

On the other hand it is not difficult to understand why τ_L does not depend appreciably on the coupling constant: when finite-volume effects first come in, that is when the wave propagating at the speed of light first starts to interfere with itself, the quantum back-reaction $\lambda\Sigma$ has settled on values of order 1, because the time $\tau_L \simeq L/2$ is much greater than the spinodal time t_1 . The slope of the linear envelope of the zero mode does depend on λ because it is fixed by the early exponential growth. Similarly, it is easy to realize that the numerical integrations of refs. [9–11] over continuum momenta correspond roughly to an effective volume much larger than ours, so that the calculated evolution remained far away from the onset of finite-volume effects.

The main limitation of the large N approximation, as far as the evolution of the widths σ_k is concerned, is in its intrinsic gaussian nature. In fact, one might envisage a scenario in which, while gaussian fluctuations stay microscopic, non-gaussian fluctuations grow in time to a macroscopic size. Therefore, in order to clarify this point and go beyond the gaussian approximation, we are going to consider, in a forthcoming work [14], a time-dependent HF approximation capable in principle of describing the dynamics of non-gaussian fluctuation of a single scalar field with ϕ^4 interaction.

Another open question concerns the connection between the minima of the effective potential and the asymptotic values for the evolution of the background, within the simplest gaussian approximation. As already pointed out in [10], a dynamical Maxwell construction occurs for the $O(N)$ model in infinite volume and at leading order in $1/N$ in case of broken symmetry, in the sense that any value of the background within the spinodal region can

be obtained as large time limit of the evolution starting from suitable initial conditions. It would be very enlightening if we could prove this “experimental” result by first principles arguments, based on CTP formalism. Furthermore, preliminary numerical evidence [14] suggests that something similar occurs also in the Hartree approximation for a single field, but a more thorough and detailed analysis is needed.

It would be interesting also to study the dynamical realization of the Goldstone paradigm, namely the asymptotic vanishing of the effective mass in the broken symmetry phases, in different models; this issue needs further study in the $2D$ case [13], where it is known that the Goldstone theorem is not valid.

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APPENDIX A: DETAILS OF THE NUMERICAL ANALYSIS

We present here the precise form of the evolution equations for the field background and the quantum mode widths, which control the out-of-equilibrium dynamics of the ϕ^4 model in finite volume at the leading order in the $1/N$ approach, as described in sections III C. We restrict here our attention to the tridimensional case.

Let us begin by noticing that each eigenvalue of the Laplacian operator in a $3D$ finite volume is of the form $k_n^2 = \left(\frac{2\pi}{L}\right)^2 n$, where n is a non-negative integer obtained as the sum of three squared integers, $n = n_x^2 + n_y^2 + n_z^2$. Then we associate a degeneracy factor g_n to each eigenvalue, representing the number of different ordered triples (n_x, n_y, n_z) yielding the same n . One may verify that g_n takes on the *continuum* value of $4\pi k^2$ in the infinite volume limit, where $k = \left(\frac{2\pi}{L}\right)^2 n$ is kept fixed when $L \rightarrow \infty$.

Now, the system of coupled ordinary differential equations is, in case of the large N approach,

$$\left[\frac{d^2}{dt^2} + M^2\right]\phi = 0, \quad \left[\frac{d^2}{dt^2} + \left(\frac{2\pi}{L}\right)^2 n + M^2\right]\sigma_n - \frac{1}{4\sigma_n^3} = 0 \quad (\text{A1})$$

where the index n ranges from 0 to \mathcal{N}^2 , $\mathcal{N} = \Lambda L/2\pi$ and $M^2(t)$ is defined by the eq. (3.16) in case of unbroken symmetry and by eq. (3.17) in case of broken symmetry. The back-reaction Σ reads, in the notations of this appendix

$$\Sigma = \frac{1}{L^D} \sum_{n=0}^{\mathcal{N}^2} g_n \sigma_n^2$$

Technically it is simpler to treat an equivalent set of equations, which are formally linear and do not contain the singular Heisenberg term $\propto \sigma_n^{-3}$. This is done by introducing the complex mode amplitudes $z_n = \sigma_n \exp(i\theta_n)$, where the phases θ_n satisfy $\sigma_n^2 \dot{\theta}_n = 1$. Then we find a discrete version of the equations studied for instance in ref [8],

$$\left[\frac{d^2}{dt^2} + \left(\frac{2\pi}{L} \right)^2 n + M^2 \right] z_n = 0, \quad \Sigma = \frac{1}{L^D} \sum_{n=0}^{\mathcal{N}^2} g_n |z_n|^2 \quad (\text{A2})$$

subject to the Wronskian condition

$$z_n \dot{\bar{z}}_n - \bar{z}_n \dot{z}_n = -i$$

One realizes that the Heisenberg term in σ_n corresponds to the centrifugal potential for the motion in the complex plane of z_n .

Let us now come back to the equations (A2). To solve these evolution equations, we have to choose suitable initial conditions respecting the Wronskian condition. In case of unbroken symmetry, once we have fixed the value of ϕ and its first time derivative at initial time, the most natural way of fixing the initial conditions for the z_n is to require that they minimize the energy at $t = 0$. We can obviously fix the arbitrary phase in such a way to have a real initial value for the complex mode functions

$$z_n(0) = \frac{1}{\sqrt{2\Omega_n}} \quad \frac{dz_n}{dt}(0) = i\sqrt{\frac{\Omega_n}{2}}$$

where $\Omega_n = \sqrt{k_n^2 + M^2(0)}$. The initial squared effective mass $M^2(t = 0)$, has to be determined self-consistently, by means of its definition (3.16).

In case of broken symmetry, the gap equation is a viable mean for fixing the initial conditions only when ϕ lies outside the spinodal region [see eq (3.21)]; otherwise, the gap equation does not admit a positive solution for the squared effective mass. In that case, we have to resort to other methods, in order to choose the initial conditions. Following the discussion presented in III C, one possible choice is to set $\sigma_k^2 = \frac{1}{2\sqrt{k^2 + |M^2|}}$ for $k^2 < |M^2|$ in eq. (3.20) and then solve the corresponding gap equation (3.20). An other acceptable choice would be to solve the gap equation (3.20), once we have set a massless spectrum for all the spinodal modes but the zero mode, which is started from an arbitrary, albeit microscopic, value.

There is actually a third possibility, that is in some sense half a way between the unbroken and broken symmetry case. We could allow for a time dependent bare mass, in such a way to simulate a sort of *cooling down* of the system. In order to do that, we could start with a unbroken symmetry bare potential (which fixes initial conditions naturally via the gap equation) and then turn to a broken symmetry one after a short interval of time. This evolution is achieved by a proper interpolation in time of the two inequivalent parameterizations of the bare mass, eqs. (3.5) and (3.10).

We looked for the influence this different choices could produce in the results and indeed they depend very little and only quantitatively from the choice of initial condition we make.

As far as the numerical algorithm is concerned, we used a 4th order Runge-Kutta algorithm to solve the coupled differential equations (A2), performing the computations in

boxes of linear size ranging from $L = 20\pi$ to $L = 400\pi$ and verifying the conservation of the Wronskian to order 10^{-5} . Typically, we have chosen values of \mathcal{N} corresponding to the UV cutoff Λ equal to small multiples of m for unbroken symmetry and of $v\sqrt{\lambda}$ for broken symmetry. In fact, the dynamics is very weakly sensitive to the presence of the ultraviolet modes, once the proper subtractions are performed. This is because only the modes inside the unstable (forbidden or spinodal) band grow exponentially fast, reaching soon non perturbative amplitudes (i.e. $\approx \lambda^{-1/2}$), while the modes lying outside the unstable band remains perturbative, contributing very little to the quantum back-reaction [9] and weakly affecting the overall dynamics. The unique precaution to take is that the initial conditions be such that the unstable band lay well within the cutoff.

FIGURES

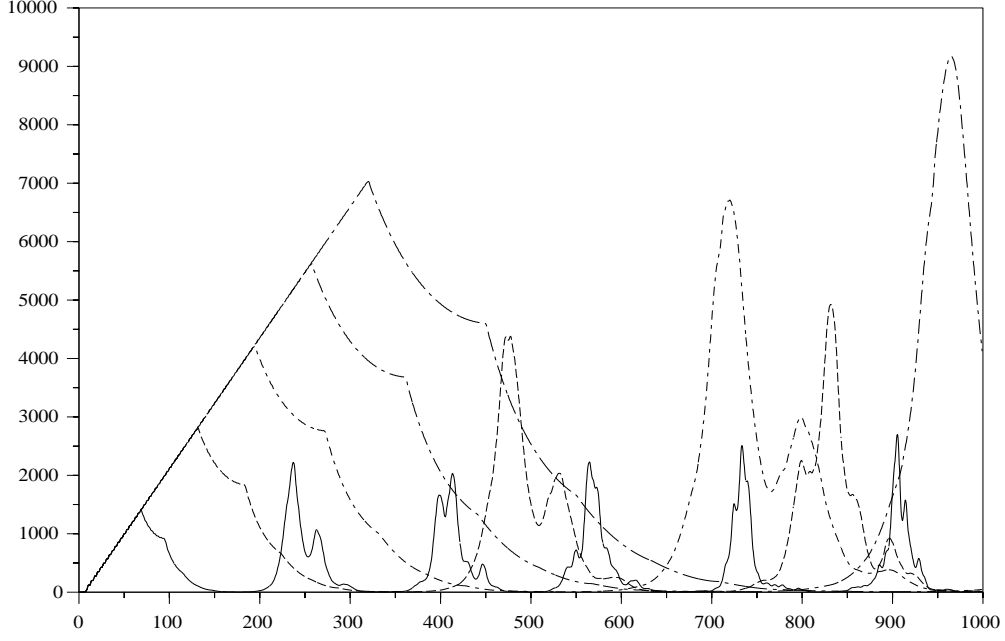


FIG. 1. Zero-mode amplitude evolution for different values of the size $\frac{L}{2\pi} = 20, 40, 60, 80, 100$, for $\lambda = 0.1$ and broken symmetry, with $\bar{\phi} = 0$.

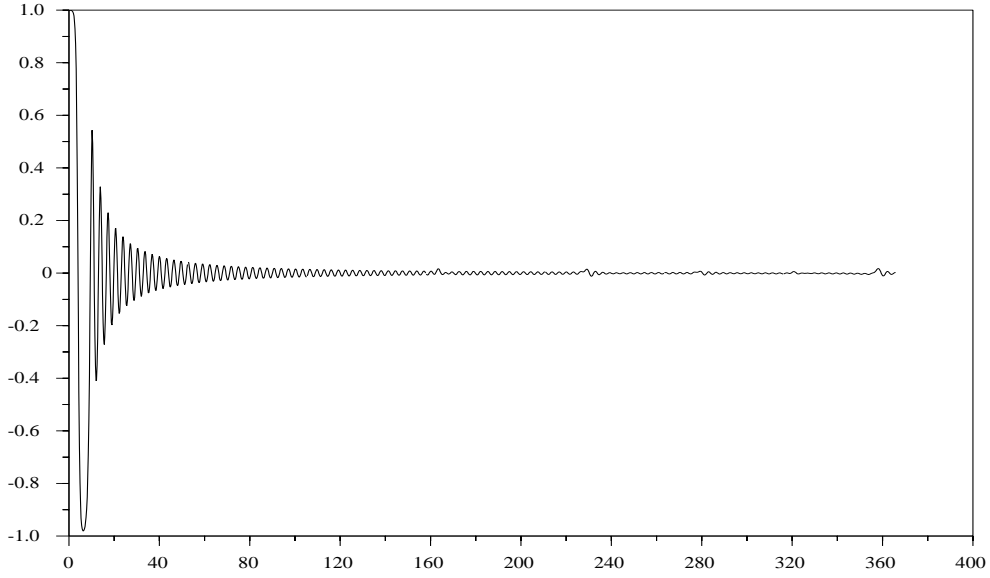


FIG. 2. Time evolution of the squared effective mass M^2 in broken symmetry, for $\frac{L}{2\pi} = 50$ and $\lambda = 0.1$.

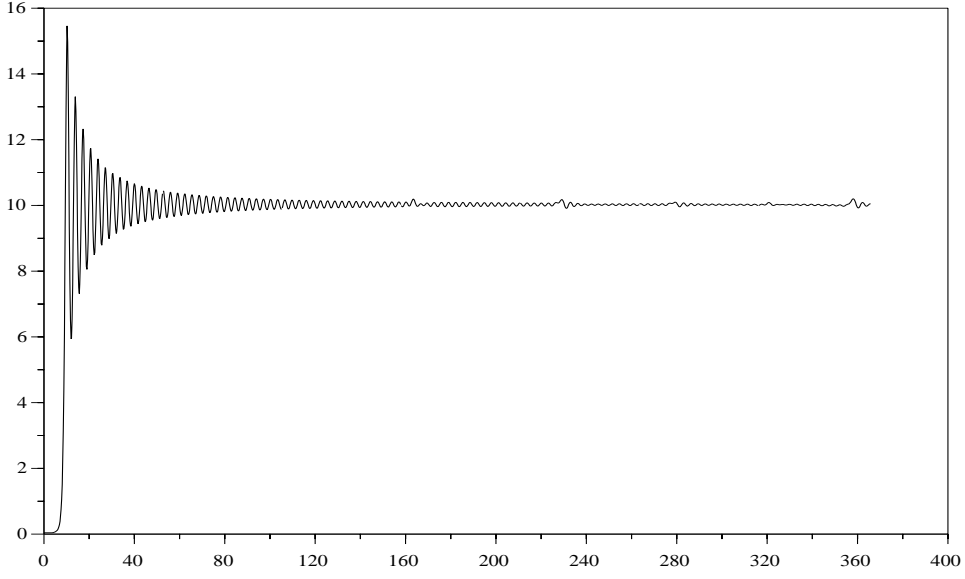


FIG. 3. The quantum back-reaction Σ , with the parameters as in Fig. 2

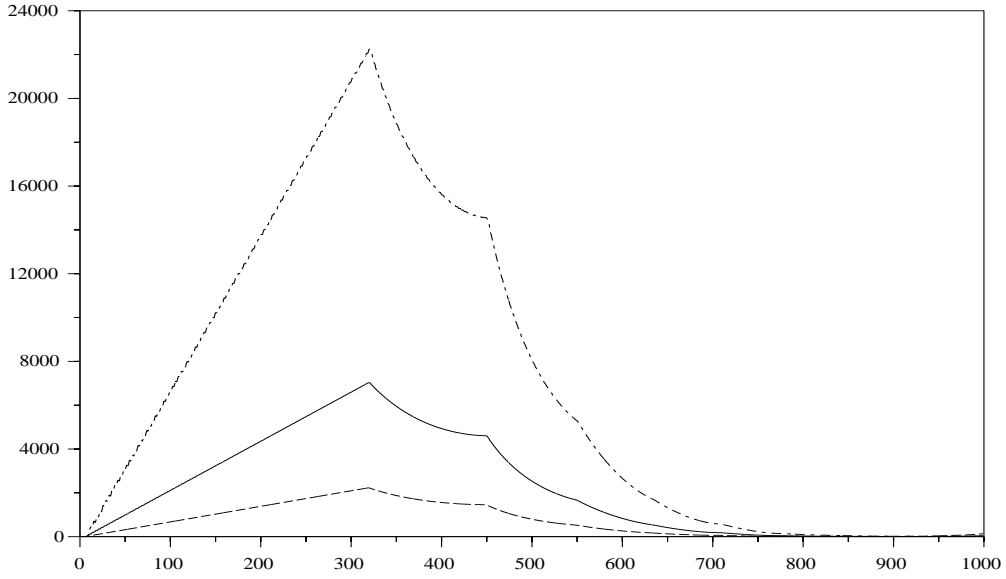


FIG. 4. Zero-mode amplitude evolution for different values of the renormalized coupling constant $\lambda = 0.01, 0.1, 1$, for $\frac{L}{2\pi} = 100$ and broken symmetry, with $\bar{\phi} = 0$.

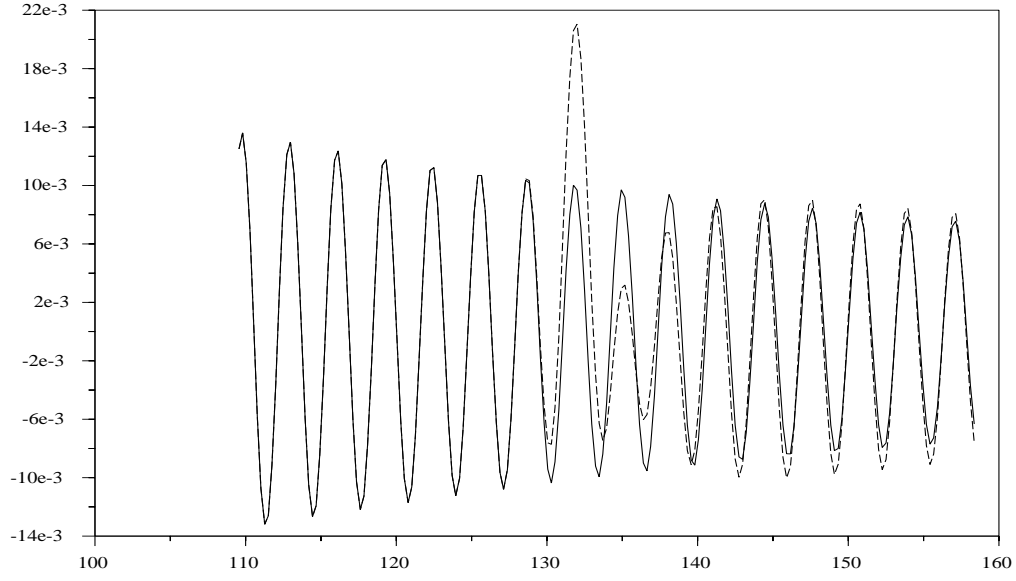


FIG. 5. Detail of M^2 near $t = \tau_L$ for $\frac{L}{2\pi} = 40$ (dotted line). The case $\frac{L}{2\pi} = 80$ is plotted for comparison (solid line).

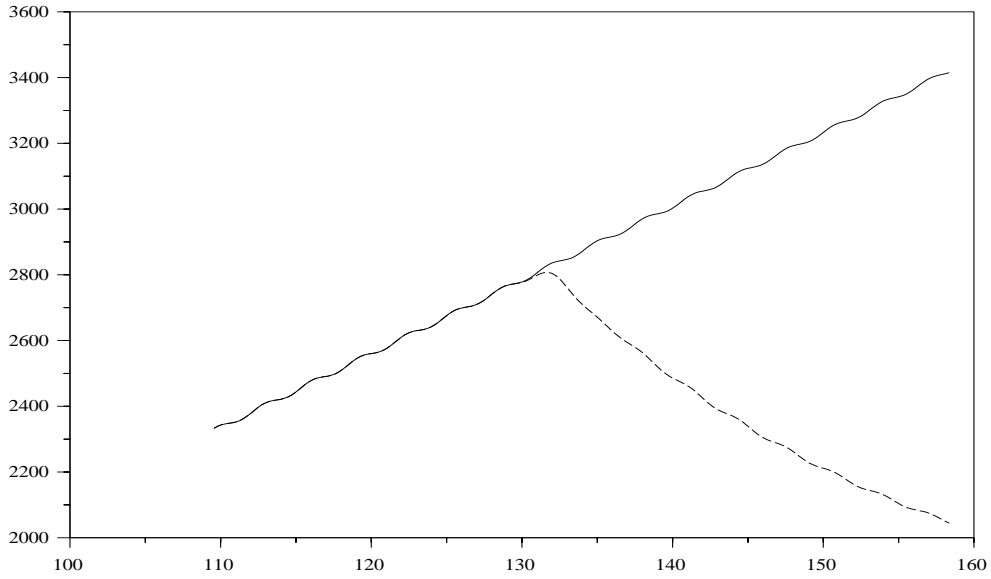


FIG. 6. Detail of σ_0 near $t = \tau_L$ for $\frac{L}{2\pi} = 40$ (dotted line). The case $\frac{L}{2\pi} = 80$ is plotted for comparison (solid line).

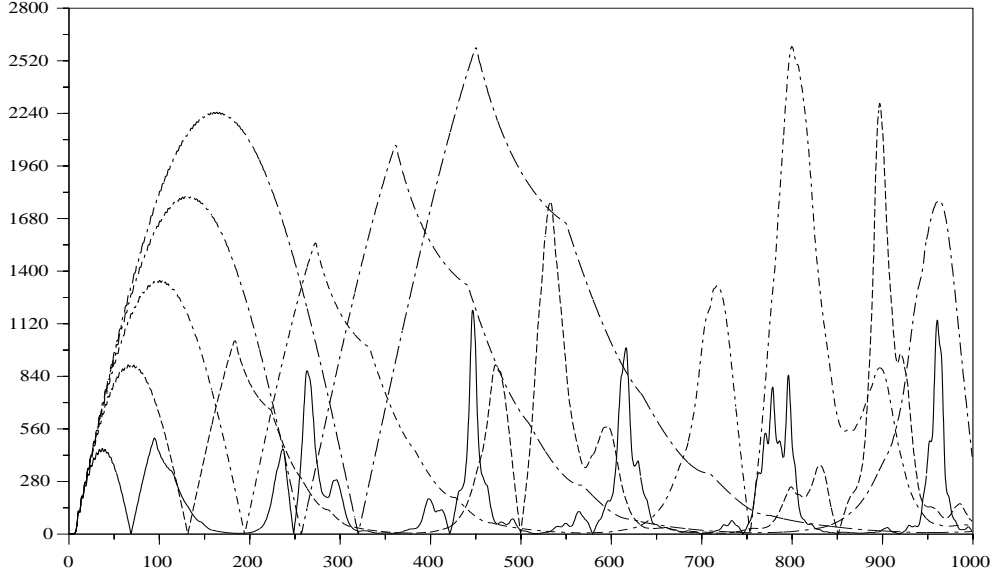


FIG. 7. *Next-to-zero mode ($k = 2\pi/L$) amplitude evolution for different values of the size $\frac{L}{2\pi} = 20, 40, 60, 80, 100$, for $\lambda = 0.1$ and broken symmetry, with $\bar{\phi} = 0$.*

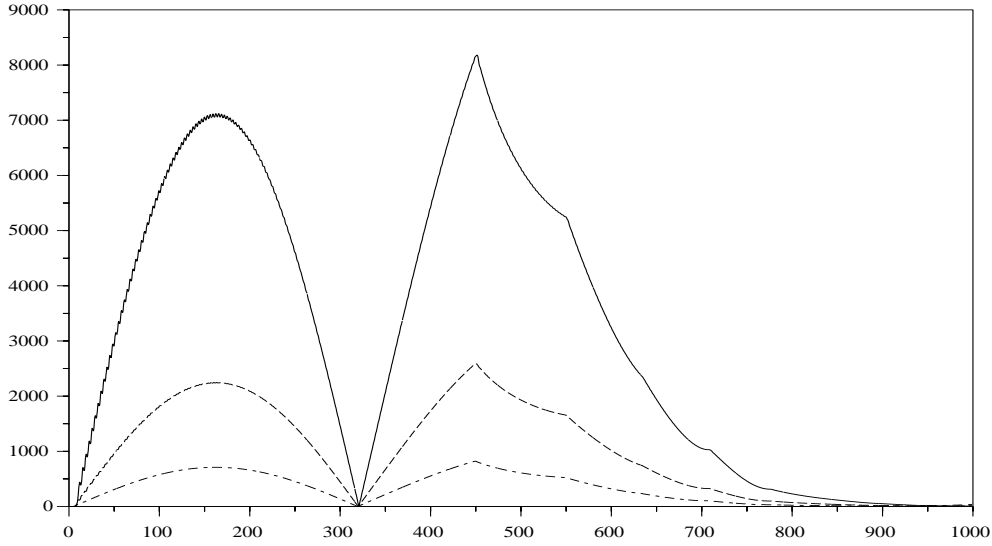


FIG. 8. *Next-to-zero mode ($k = 2\pi/L$) amplitude evolution for different values of the renormalized coupling constant $\lambda = 0.01, 0.1, 1$, for $\frac{L}{2\pi} = 100$ and broken symmetry, with $\bar{\phi} = 0$.*

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